Eur. Phys. J. B **19**, 25–36 (2001)

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

Integrable correlated electron model with next-nearest-neighbour interactions

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Received 13 September 2000

Abstract. The exactly solvable model of supersymmetric $t-J$ chains (STJC) of correlated electrons with next-nearest-neighbour (NNN) interactions is proposed and studied. The model with interactions between nearest neighbours and NNN interactions in one chain can also be considered as a two-chain model with zigzag-like coupling between the chains. The NNN interaction (coupling between chains) causes the onset of additional Dirac seas for low-lying charge and/or spin excitations. These Dirac seas change the low-energy (conformal) behavior of the model. The filling of those seas depends on the values of the NNN coupling (interactions between chains), external magnetic field and applied voltage. We identify the new ground state phases which appear due to the NNN as incommensurate ones. The NNN coupling in the incommensurate phases induces spontaneous magnetization and/or spontaneous filling of the Dirac sea for charge excitations ("spontaneous charge ordering"). The onset of this order implies a first order quantum phase transition driven by the field with hysteresis phenomena.

PACS. 71.10.Hf Non-Fermi-liquid ground states, electron phase diagrams and phase transitions in model systems – 71.27.+a Strongly correlated electron systems; heavy fermions – 71.10.Fd Lattice fermion models (Hubbard model, etc.) – 71.10.Pm Fermions in reduced dimensions (anyons, composite fermions, Luttinger liquid, etc.)

1 Introduction

There has been recently considerable interest in low-dimensional quantum correlated spin and electron systems. These systems, especially one-dimensional (1D), manifest specific features of, *e.g.*, magnetic behavior at low temperatures, which are absent for the standard, conventional 3D magnetic systems. Quasi-1D systems usually manifest 1D behavior for temperatures higher than the temperature of the 3D magnetic ordering, but lower than the maximal characteristic energy of the 1D interaction between particles. The origin of such specific features is the enhancement of quantum and thermal fluctuations of the 1D systems due to the peculiarities of the 1D density of states together with the quantum nature of electrons. These fluctuations usually destroy any ordering of 1D systems at nonzero temperatures and most often in the ground state (with only few exceptions). However correlations usually exhibit power-law decays in the ground state of interacting 1D systems. This is why such powerful methods of theoretical physics as perturbation theories and mean-field approximation (of any kind) are hardly applicable to strongly correlated 1D systems. Thus low dimensional correlated electron and spin systems are perhaps the best known examples in which non-perturbative methods like the renormalization group (RG) theory, Bethe's ansatz, bosonization, conformal field theory, etc. have manifested their advantages. Moreover, during the last decade a large number of new quasi 1D spin and correlated electron compounds were created and experimentally studied. These compounds manifest properties of a single or several quantum chains weakly coupled to each other at low temperatures [1,2]. This class of compounds will probably provide new information about the transition from 1D to 2D in quantum many-body physics. It is very important, because 2D quantum many-body physics has been a challenge for both theorists and experimentalists since the beginning of the study of low dimensional quantum systems. On the other hand, the advantage of theoretical 1D studies is the possibility

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to obtain exact solutions by using non-perturbative methods, which are difficult to apply for higher-dimensional quantum many-body models. The results of exact calculations of 1D models can serve as a testing ground for the use of perturbative and numerical methods in more realistic situations.

Recently several exactly solvable *spin* models [3–8] were proposed, in which zigzag-like interactions between two quantum spin chains were exactly studied using the Bethe ansatz technique [9]. This method is widely known by now, see, *e.g.* [10]. The Bethe ansatz method permits to calculate exactly static characteristics of quantum many-body systems. These results should be used for more realistic systems such as coupled chains of strongly correlated electrons. However it is not obvious how the interactions between the chains will modify the known answers for single correlated electron chains [11]. Mean-field like approximations for the inter-chain couplings are not legitimate, because the mean field approach in any version already implies the existence of the (sometimes hidden) order parameter. Also, it is unfortunately not clear whether numerical simulations, which can be directly applied to quantum many-body systems of rather small sizes (say, at most several tens of sites) describe well the properties of the real systems, in which, even in quasi-1D, the number of sites is at least of order of 10^8 or higher. On the other hand, it must be admitted that some features of exactly solvable 1D models are far from being observed experimentally. However these non-realistic features of the 1D models are known and simple to recognize. The behavior of multi-chain many-body quantum systems in an external magnetic field is especially interesting, see for instance [12–14], because of the possibility of experimental observations due to recent progress in high magnetic field measurements. On the other hand, they are important due to very interesting, theoretically predictable effects, which are possible to be realized in experiments, like phase transitions in an external magnetic field. However, several important issues are far from being solved in quantum two-chain models. For example, it is not clear how the changes of an external applied voltage (which plays the role of a chemical potential) will affect properties of two coupled 1D correlated electron chains. Here we point out recent experimental observations, *e.g.*, in α'-NaV_2O_5 [15] and CaV_3O_7 [16]. They manifest the special behavior of V magnetic ions, namely the "spontaneous charge ordering". This means that at sufficiently high temperature the valence of V ions is homogeneous, *i.e.*, equal to 4.5. However, below a certain critical temperature some V ions reveal a valence close to 4, while others are almost 5-valent [17]. Charge ordering was also observed in the heavy fermion compound Yb_4As_3 [18]. It is also known that for a large class of compounds, so-called magnetic Jahn-Teller systems [19,20], the properties can be described by a model which manifests strong Hubbard repulsion and (similar for simple lattices) *exchange* interactions between two bands of electrons. In the limit of the Hubbard coupling being much larger than the hopping integral, the second-order perturbation theory gives rise to the Hamiltonian of the $t-J$ model [21]. There, the electrons hop between neighboring sites of the lattice and manifest the spin-exchange interaction when being nearest neighbors.

In this paper we propose a correlated electron model which can be exactly solved by means of the algebraic Bethe ansatz. We study the supersymmetric $t-J$ chains (STJC) with NNN exchange coupling and NNN hopping. It can also be considered as two STJCs coupled to each other by zigzag-like two-particle interaction and by four-particle interactions. Supersymmetry (in contrast to the standard $t-J$ model [21]) appears due to fixing the value of the exchange constant to $J = 2$ (in units of the hopping integral). This value allows to obtain an exact Bethe ansatz solution for a single STJC [22–24]. We shall show that in the ground state our model exhibits several quantum phase transitions (governed by an external magnetic field and by the change of band-filling, due to an external voltage). We determine them as the *commensurate-incommensurate* phase transitions due to the NNN couplings between correlated electrons. We shall also show that the *spontaneous magnetization and spontaneous filling of the charge band* appear with first order phase transitions. To the best of our knowledge, this is the first exact study in which the spontaneous "charge ordering" and magnetization appear.

The paper is organized as follows: after the Introduction, in Section 2 we briefly describe the scheme of the algebraic Bethe ansatz for the STSC with NNN interactions (two-chain model). The ground state properties (with conformal field theory analysis of the low-energy behavior) of the model are presented in Section 3. Concluding remarks follow in Section 4.

2 Algebraic Bethe ansatz for coupled chains

2.1 Algebraic relations

The scheme of the quantum inverse scattering method [10] (algebraic Bethe ansatz) for the STJC with only nearest-neighbour couplings was presented in [25]. Here we briefly sketch the main steps of that method for the model with NNN interactions, which we want to study. One starts with the R-matrix which depends on the spectral parameter u. For the STJC it has the form:

$$R(u) = [1 - a(u)]\hat{I} + a(u)\hat{P} \,, \tag{1}$$

where $a(u) = u/(u+\mathrm{i})$, $\hat{I}$ is the identity operator and $\hat{P}$ is the graded permutation operator [25]. "Grading" is connected with the supersymmetry. The Hamiltonian of the STJC can be represented by a quadratic form of fermionic and bosonic currents (generators of the $spl(2,1)$ superalgebra). The coupling constants for these currents are equal due to the supersymmetry [25]. Notice though different signs in the nontrivial commutation relations between the bosonic and fermionic generators. These signs are positive for bosons and negative for fermions. In this grading one can write $\hat{I}^{\gamma,\nu}_{\alpha,\beta} = \delta_{\alpha,\beta}\delta_{\gamma,\nu}$ and $\hat{P}^{\gamma,\nu}_{\alpha,\beta} = (-1)^{\epsilon_\nu \epsilon_\beta}\delta_{\alpha,\nu}\delta_{\gamma,\beta}$,

where ϵ_α equals 1 for fermions and 0 for bosons. The R-matrices satisfy the Yang-Baxter relation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \ , \tag{2}$$

where the lower indices denote the Hilbert spaces in which the R matrix acts. We may consider spaces 1 and 2 as "quantum spaces" and 3 as "auxiliary space". The L-operators on site l (whose quantum space corresponds to the Hilbert space of the lth site of the chain), $L(u) = \hat{P}R(u)$, also satisfy the Yang-Baxter relations. These L-operators are direct products of the two-particle scattering matrix acting on the lth site of the chain and the auxiliary space with unity operators for the Hilbert spaces of the other sites. We define the monodromy matrix as the ordered product of the L-operators

$$T_L(u) = L_L(u)L_{L-1}(u)\dots L_1(u) \ . \tag{3}$$

By construction the monodromy matrices satisfy the Yang-Baxter equation with the R-matrix. The supertrace (with the positive contributions from the bosonic degrees of freedom and negative ones from the fermionic counterpart) of the monodromy matrix $\tau(u) = s\mathrm{Tr}[T_L(u)]$ is the transfer matrix of the associated 2D statistical vertex problem. As a consequence of the Yang-Baxter relations for monodromies, transfer matrices with different spectral parameters mutually commute. This means that the transfer matrix can serve as a generating functional of the (infinite) number of conserved quantities, like the Hamiltonian, the operator of the total momentum, etc. For example, the Hamiltonian of the STJC with nearest-neighbour couplings is usually constructed by taking the first derivative of the logarithm of the transfer matrix at zero spectral parameter

$$\mathcal{H}_2 = -\mathrm{i}\frac{\partial \ln\tau(u)}{\partial u}|_{u=0} - L = -\sum_{l=1}^{L} P_{l,l+1} \ , \tag{4}$$

where $P_{l,l+1}$ is the graded permutation operator for sites l and $l+1$. It has the form

$$P_{l,l+1} = -\sum_{\alpha,\beta=1}^{9} K_{\alpha,\beta} J_l^\alpha J_{l+1}^\beta \ , \tag{5}$$

with $J^\alpha = \sum_j J_j^\alpha$ being four fermionic and five bosonic operators of the conserved currents (generators of the superalgebra $spl(2,1)$) [25]. The coupling coefficients are (graded) supertraces (with positive terms for bosonic and negative terms for fermionic parts) of the total conserved currents, $K_{\alpha,\beta} = s\mathrm{Tr}(J^\alpha J^\beta)$. For convenience we write down the non-vanishing elements of this matrix, namely $K_{1,2} = K_{2,1} = 2K_{3,3} = -K_{4,5} = K_{5,4} = -K_{6,7} = K_{7,6} = -2K_{8,8} = K_{9,9} = -1$. Bosonic generators are the unity operator ($\alpha = 9$, we keep the notations of [25]), the operator of the total number of electrons $\hat{N}$, ($\alpha = 8$), and three operators of the projections of the total spin, $S^{\pm,z}$ ($\alpha = 1,2,3$), respectively. They form $U(1)$ and $SU(2)$ subalgebras ($[S^z, S^\pm] = \pm S^\pm$, $[S^+, S^-] = 2S^z$) of $spl(2,1)$. The fermion currents $Q_{1,2}^\pm$ ($\alpha = 4 \div 7$) satisfy the anticommutation relations (see, *e.g.*, [26])

$$\{Q_1^\pm, Q_2^\pm\} = \pm\frac{S^\pm}{2} \ , \quad \{Q_1^\pm, Q_2^\mp\} = \pm\frac{-S^z \pm \hat{N}}{2} \ \cdot \tag{6}$$

with other mutual anticommutators being zero. They satisfy the commutation relations with the bosonic generators

$$\begin{aligned} &[S^z, Q_l^\pm] = \pm\frac{Q_l^\pm}{2} \ , \quad [\hat{N}, Q_l^\pm] = (-1)^{l+1}\frac{Q_l^\pm}{2} \ , \\ &[S^\mp, Q_l^\pm] = Q_l^\mp \ , \quad [S^\pm, Q_l^\pm] = 0 \ , \end{aligned} \tag{7}$$

with $l = 1,2$. This superalgebra can be written in compact form as

$$\begin{aligned} [J_l^\alpha, J_l^\beta\} &\equiv J_l^\alpha J_l^\beta - (-1)^{\epsilon_\alpha\epsilon_\beta} J_l^\beta J_l^\alpha \\ &= f_\gamma^{\alpha\beta} J_l^\gamma . \end{aligned} \tag{8}$$

In the basis where $\hat{N}$, $\mathbf{S}^2$ and S^z are diagonal, the non-vanishing matrix elements of $Q_{1,2}^\pm$ are

$$\begin{aligned} &\langle S+\frac{1}{2}, S-\frac{1}{2}, \sigma\pm\frac{1}{2}|Q_1^\pm|S,S,\sigma\rangle = \pm\sqrt{\frac{S\mp\sigma}{2}} \ , \\ &\langle S,S,\sigma|Q_2^\pm|S+\frac{1}{2}, S-\frac{1}{2}, \sigma\mp\frac{1}{2}\rangle = \sqrt{\frac{S\pm\sigma}{2}} \ \cdot \end{aligned} \tag{9}$$

We choose $S = \frac{1}{2}$ and can express the operators of the conserved currents in terms of the standard electron creation and annihilation operators as $\hat{N} = \sum_j (n_{j,+} + n_{j,-})$, $2S^z = \sum_j (n_{j,+} - n_{j,-})$, $S^\pm = \sum_j c_{j,\mp}^\dagger c_{j,\pm}$, $Q_1^+ = \sum_j (1 - n_{j,-})c_{j,+}^\dagger$, $Q_2^+ = \sum_j (1 - n_{j,+})c_{j,-}$, and $Q_{1,2}^- = (Q_{1,2}^+)^+$. Here $n_{j,\pm} = c_{j,\pm}^\dagger c_{j,\pm}$, and $c_{j,\pm}^\dagger$ as usual create an electron with z-projection of spin $\pm\frac{1}{2}$ at site j. The multipliers $(1 - n_{j,\mp})$ of fermionic conserved currents exclude double occupations of each site, as it must be for the $t-J$ model. These generators can be easily expressed in terms of the Hubbard operators, see, *e.g.*, [27].

Now we propose to study the STJC with nearest and NNN interactions (or in other words, two STJCs coupled with zigzag-like two-particle interactions and four-particle interactions to each other). For this purpose we construct the Hamiltonian which consists of two parts, $\mathcal{H} = \mathcal{H}_2 + A\mathcal{H}_4$. Here $\mathcal{H}_4$ is the fourth "conservation law" of the associated statistical problem. It is the third logarithmic derivative of the transfer matrix of the associated statistical model taken at zero value of the spectral parameter. This part of the Hamiltonian, which contains the NNN interactions, can be written as [25]

$$\begin{aligned} \mathcal{H}_4 = &-2\mathcal{H}_2 + \sum_{l=1}^{L} \Pi_{l-1,l+1} \\ &- 2\sum_{l=1}^{L} K_{\mu\nu}K_{\alpha\beta}K_{\gamma\delta} f_\epsilon^{\beta\gamma} f_\omega^{\delta\mu} J_{l-1}^\alpha J_l^\epsilon J_{l+1}^\omega J_{l+2}^\nu \end{aligned} \tag{10}$$

with $\Pi_{l-1,l+1} = P_{l-1,l}P_{l,l+1}P_{l-1,l}$ being the graded permutation operator between sites $l-1$ and $l+1$. Hence

the total Hamiltonian can be considered as one for electrons of one STJC with NNN interactions (proportional to A), or for electrons of two STJCs with intra-chain coupling constants A and inter-chain zigzag coupling (equal to $1-2A$ here). By construction, the model is exactly solvable. There exists additional four-site interaction between electrons. A similar four-site coupling has been already discussed in the literature in connection with the spin-ladder systems [29] and with magnetic co-operative Jahn-Teller systems [20]. Changing the value of the coefficient A we can study all possible situations from the limit of a single STJC (with only nearest neighbor interactions), $A=0$, to the limit of two decoupled STJCs (or one chain with the only NNN interactions) for $A=\frac{1}{2}$ (notice, though, that four-site interactions survive in this case).

2.2 Problem of eigenvalues

Now, our goal is to find the eigenfunctions and eigenvalues of the transfer matrix. Then the eigenvalues for the conservation laws will be calculated from the eigenvalue of the transfer matrix. For definiteness we will work in the Hilbert quantum space of the kth site, in which $\epsilon_1=\epsilon_2=1$ and $\epsilon_3=0$, *i.e.*, FFB grading (representing spin up and spin down electron states (fermionic) and a hole state (bosonic) at each site). We choose the vacuum state $|0\rangle_k$ as purely bosonic. One can represent the monodromy matrix $T_L(u)$ as the 3×3 operator valued matrix $\hat{A}_{ij}$ with supertrace $\tau(u)=\hat{A}_{33}-\hat{A}_{11}-\hat{A}_{22}$. The action of the monodromy matrix on the vacuum state $|0\rangle=\otimes_{k=1}^{L}|0\rangle_k$ has triangular form (with the elements 12, 13, 21 and 23 of $\hat{A}_{i,j}$ being zero). Other non-diagonal elements can serve as "creation operators" with respect to our vacuum state, *i.e.*, we can use the following form of eigenstates $|\{u'\}_{j=1}^{N}|F\rangle=\prod_{j=1}^{N}C_{a_j}(u'_j)|0\rangle F^{a_N\cdots a_1}$, where $C_{a_j}=\hat{A}_{3,a_j}$ and a_j runs over 1 and 2. The commutation relations between the elements of the monodromy matrix follow from the Yang-Baxter relations. Acting with the transfer matrix on such a state produces the state multiplied by the eigenvalue (as a function of the spectral parameter u and *charge rapidities* u'_j) and some other "unwanted" terms, which we want to cancel. The cancellation of the unwanted terms yields N conditions (Bethe ansatz equations) for the parameters u'_j.

The commutation relations for the "creation operators"

$$C_{a_1}(u'_1)C_{a_2}(u'_2)=r_{b_2a_1}^{b_1a_2}(u'_1-u'_2)C_{b_2}(u'_2)C_{b_1}(u'_1) \quad (11)$$

involve some additional 4×4 "nesting" r-matrix, which describes the spin degrees of freedom of our system. This nesting matrix has the form

$$r(u)=[1-a(u)]\hat{I}^s+a(u)\hat{P}^s\ , \quad (12)$$

with $\hat{I}^s$ and $\hat{P}^s$ being 4×4 unity and graded ($\epsilon_{1,2}=1$) permutation operators. Those r-matrices also mutually satisfy the Yang-Baxter relations. For this nesting problem we construct the eigenvalues for the nested transfer matrix with $F^{a_N\cdots a_1}$ serving as the basis for the nested eigenstates $|\{\lambda\}_{\alpha=1}^{M}\rangle=\prod_{\alpha=1}^{M}C^s(\lambda_\alpha)|0\rangle_{a_N\ldots a_1}=F^{a_N\ldots a_1}$ in a similar way as above. The *spin rapidities* λ_α are determined from the condition of the cancellation of the "unwanted" terms in the eigenvector problem for the nesting transfer matrix (in the spin subspace). Combining all the effects we write down the eigenvalue for the transfer matrix of the STJC as

$$\begin{aligned}\Lambda(u)=&\prod_{j=1}^{N}a^{-1}(u'_j-u)-a^L(u)\\ &\times\Bigg(\prod_{j=1}^{N}a^{-1}(u'_j-u)\prod_{\alpha=1}^{M}a^{-1}(u-\lambda_\alpha)\\ &+\prod_{\alpha=1}^{M}a^{-1}(\lambda_\alpha-u)\Bigg)\ . \end{aligned}\quad (13)$$

The Bethe ansatz equations for charge and spin rapidities have the form

$$\begin{aligned}\prod_{k=1}^{N}a(u'_j-\lambda_\alpha)&=\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{M}\frac{a(\lambda_\beta-\lambda_\alpha)}{a(\lambda_\alpha-\lambda_\beta)}\ ,\\ a(u'_j)^L&=\prod_{\alpha=1}^{M}a(u'_j-\lambda_\alpha)\ . \end{aligned}\quad (14)$$

The energy of the nearest-neighboring part of the total Hamiltonian ($\mathcal{H}_2$) is determined as the first logarithmic derivative, and the energy of the next-nearest-neighbors' part ($\mathcal{H}_4$) is determined as the third logarithmic derivative of the eigenvalue of the transfer matrix (13) at $u=0$. It turns out that those parts of the total Hamiltonian commute mutually and with the transfer matrix.

Let us consider the Bethe ansatz equations for periodic boundary conditions for the sets of spin $\{\lambda_\alpha\}_{\alpha=1}^{M}$ and charge $\{p_j\}_{j=1}^{N}$ rapidities (note that for continuum models charge rapidities coincide with quasimomenta of charge excitations). N and M denote the number of electrons and the number of "down spins", respectively. For convenience, we use the shift $u'_j=p_j-\frac{\mathrm{i}}{2}$ defining p_j. The structure of the Bethe ansatz equations for some other strongly correlated electron models [11] is similar to equations (14) because the corresponding scattering processes in the spin and charge subspaces possess similar symmetries. The Bethe ansatz equations are:

$$\begin{aligned}\prod_{j=1}^{N}e_1(\lambda_\alpha-p_j)&=-\prod_{\beta=1}^{M}e_2(\lambda_\alpha-\lambda_\beta)\\ e_1^L(p_j)&=\prod_{\beta=1}^{M}e_1(p_j-\lambda_\beta)\ , \end{aligned}\quad (15)$$

where L is the number of sites, and $e_n(x)=(2x+\mathrm{i}n)/(2x-\mathrm{i}n)$. In fact, the Bethe ansatz equations are the quantization conditions for the rapidities. In the absence of interactions and inhomogeneities in the system they coincide

with the well-known ones for the quasimomenta of the free particles in a box of length L. These rapidities parametrize the eigenstates and eigenvalues of the Bethe ansatz-solvable model in a well established way [10]. The energy and the magnetization of the STJC with nearest and NNN interactions have the simple form:

$$E = \sum_{j=1}^{N} \left(1 - A\frac{\partial^2}{\partial p_j^2}\right) \frac{4}{4p_j^2+1} - L \ ;$$
$$M^z = \frac{N}{2} - M \ . \tag{16}$$

Equations (15) are written for the STJC in the Lai-Schlottmann's form (FFB grading) [22,24,25]. However, for some purposes (see below) it will be more convenient to use Sutherland's form (BFF grading) [23]. We re-write equations (15) largely following [28,25]:

$$e_1^L(u_j) = -\prod_{k=1}^{N^h+M} e_2(u_j - u_k) \prod_{\beta=1}^{N^h} e_1^{-1}(u_j - \nu_\beta)$$
$$1 = \prod_{k=1}^{N^h+M} e_1(\nu_\alpha - u_k) \ , \tag{17}$$

where $N^h = L - N$ is the number of holes (non-occupied sites), and

$$E = -\sum_{j=1}^{N^h+M} \left(1 - A\frac{\partial^2}{\partial u_j^2}\right) \frac{4}{u_j^2+4} + L \ ;$$
$$M^z = \frac{L-N^h}{2} - M \ . \tag{18}$$

It is easy to show that the two forms, equations (15) and equations (17), are equivalent. For this purpose one can consider the second set of the Bethe ansatz equations in Sutherland's form (17) as the roots of some polynomial $P(\nu_\alpha) = 0$ with

$$P(x) = \prod_{k=1}^{N^h+M} (x - u_k - \frac{\mathrm{i}}{2}) - \prod_{k=1}^{N^h+M} (x - u_k + \frac{\mathrm{i}}{2}) \ . \tag{19}$$

We separate the first N^h roots ν_α of the $N^h + M$ roots of $P(x)$ and label the remaining M roots by λ_α. Then we have the factorization

$$P(x) = \text{const.} \times \prod_{\alpha=1}^{M} (x - \lambda_\alpha) \prod_{\beta=1}^{N_h} (x - \nu_\beta) \tag{20}$$

from which follows

$$\prod_{\alpha=1}^{M} e_1(u_j - \lambda_\alpha) \prod_{\beta=1}^{N^h} e_1(u_j - \nu_\beta) =$$
$$\frac{P(u_j + \mathrm{i}/2)}{P(u_j - \mathrm{i}/2)} = -\prod_{k=1}^{N^h+M} e_2(u_j - u_k) \ . \tag{21}$$

Then using this relation and the first set of equations (17) we obtain the second set of equations (15), with $u_j = p_j$.

Next, the second equation of (15) can be re-written as $Q(p_j) = 0$ with the definition

$$Q(x) = (x + \frac{\mathrm{i}}{2})^L \prod_{\beta=1}^{M} (x - \lambda_\beta - \frac{\mathrm{i}}{2}) - (x - \frac{\mathrm{i}}{2})^L \prod_{\beta=1}^{M} (x - \lambda_\beta + \frac{\mathrm{i}}{2}) \ . \tag{22}$$

As above, separating first the N roots p_j of this polynomial and labelling the remaining $N^h + M$ roots by u_k we obtain the factorization

$$Q(x) = \text{const.} \times \prod_{j=1}^{N} (x - p_j) \prod_{k=1}^{N_h+M} (x - u_k). \tag{23}$$

From this we get

$$\prod_{j=1}^{N} e_1(\lambda_\alpha - p_j) \prod_{k=1}^{N^h+M} e_1(\lambda_\alpha - u_k) =$$
$$\frac{Q(\lambda_\alpha + \mathrm{i}/2)}{Q(\lambda_\alpha - \mathrm{i}/2)} = -\prod_{\beta=1}^{M} e_2(\lambda_\alpha - \lambda_\beta) \ . \tag{24}$$

Together with the second set of equations (17) for λ_α in place of ν_α it gives the first set of equations (15).

3 Ground state and low-temperature properties

3.1 Ground state integral equations

Let us proceed along the well-known lines [10] to obtain the characteristics of the eigenstates of our STJC with NNN couplings in the thermodynamic limit, *i.e.*, for $L, N, M \to \infty$ but with their ratios fixed. Instead of the huge set of transcendental algebraic equations for the rapidities one obtains a system of integral equations for the energies of excitations and/or for the densities of the distributions of rapidities. (As usual, we take the logarithms of equations (15, 17) and introduce the distributions of the rapidities $p \leftrightarrow p_j$, $\lambda \leftrightarrow \lambda_\alpha$, $u \leftrightarrow u_j$ and $\nu \leftrightarrow \nu_\beta$ as functions of the distributions of the logarithm's branch numbers in the limit $L, N, M \to \infty$.) In the ground state this set of integral equations is finite. (For nonzero temperatures it is infinite in the framework of the "string hypothesis", however such a set can be written in a finite form in the so-called "quantum transfer matrix" approach [30].) There are two integral equations for dressed densities and two for dressed energies. Each of these equations corresponds to the charge and spin degrees of freedom of the system. As usual, the interaction "dresses" the "bare" functions corresponding to the free or "driving" terms of the integral equations. In Lai-Schlottmann's form

these equations read (notice, that in Lai-Schlottmann's construction the ground state pertains to the filling up of the Dirac seas for unbound electron excitations, carrying spin $\frac{1}{2}$ and charge $-e$, and singlet pairs corresponding to solutions to Eqs. (15) with $p_\alpha = \lambda_\alpha \pm \mathrm{i}/2$ [24] carrying zero spin and charge $-2e$):

$$\begin{aligned} \rho_{\rm c} &= a_2 - a_2 \star \rho_{\rm c} - a_1 \star \rho_{\rm s} \ , \\ \rho_{\rm s} &= a_1 - a_1 \star \rho_{\rm c} \end{aligned} \tag{25}$$

and

$$\begin{aligned} \varepsilon_{\rm c}(\lambda) &= -2\mu + 2\pi \left[1 - A\frac{\partial^2}{\partial \lambda^2}\right] a_2(\lambda) \\ &\quad -(a_2 \star \varepsilon_{\rm c})(\lambda) - (a_1 \star \varepsilon_{\rm s})(\lambda) \ , \\ \varepsilon_{\rm s}(p) &= -\mu - \frac{H}{2} + 2\pi \left[1 - A\frac{\partial^2}{\partial p^2}\right] a_1(p) \\ &\quad -(a_1 \star \varepsilon_{\rm c})(p) \end{aligned} \tag{26}$$

with $\rho_{\rm c}(\lambda)$ ($\rho_{\rm s}(p)$) and $\varepsilon_{\rm c}(\lambda)$ ($\varepsilon_{\rm s}(p)$) being dressed densities (distributions of rapidities) and dressed energies for the charge (spin) low-lying excitations. (The approach with dressed energies of low-lying excitations is complementary to the densities' approach [10]. In such an approach the formation of the ground state – the search of the Dirac seas' filling for the model – can be managed in the most natural way.) In Lai-Schlottmann's formulation charge low-lying excitations are spin-singlet pairs and spin excitations are unbound electron excitations [24]. The functions $a_n(x)$ are the Fourier transforms of $\exp(-n|\omega|/2)$ yielding algebraic functions, μ is the Lagrange multiplier (chemical potential or an external applied voltage), H is an external magnetic field, and $\star$ denotes the convolution $a \star b = \int a(x-y)b(y)\mathrm{d}y$. The limits of integrations are determined from the conditions for the dressed energies to be negative. Hence they pertain to the filling of the Dirac seas (*i.e.*, in the ground state all possible states with negative energies are occupied). In Sutherland's representation the integral equations have the form

$$\begin{aligned} \rho_{\rm sp} &= a_1 - a_2 \star \rho_{\rm sp} + a_1 \star \rho_{\rm h} \ , \\ \rho_{\rm h} &= a_1 \star \rho_{\rm sp} \end{aligned} \tag{27}$$

and

$$\begin{aligned} \varepsilon_{\rm sp}(u) &= -2\pi \left[1 - A\frac{\partial^2}{\partial u^2}\right] a_1(u) + H \\ &\quad -(a_2 \star \varepsilon_{\rm sp})(u) + (a_1 \star \varepsilon_{\rm h})(u) \ , \\ \varepsilon_{\rm h}(\nu) &= \mu - \frac{H}{2} + (a_1 \star \varepsilon_{\rm sp})(\nu) \ , \end{aligned} \tag{28}$$

with $\rho_{\rm h}(\nu)$ ($\rho_{\rm sp}(u)$) and $\varepsilon_{\rm h}(\nu)$ ($\varepsilon_{\rm sp}(u)$) being dressed densities and dressed energies for the holon (spinon) low-lying excitations. Here (in Sutherland's formulation) spin excitations are spinons which carry zero charge and spin $\frac{1}{2}$, and holons which carry charge e. We emphasize again that Lai-Schlottmann's and Sutherland's representations produce the same answers (but in a different form).

The integrals over the densities of rapidities (distribution functions) determine the number of electrons in the system and the total magnetization. In Lai-Schlottmann's form they are:

$$\begin{aligned} \frac{N}{L} &= \int \mathrm{d}p \rho_{\rm s}(p) + 2\int \mathrm{d}\lambda \rho_{\rm c}(\lambda) \ ; \\ \frac{M^z}{L} &= \frac{1}{2}\int \mathrm{d}p \rho_{\rm s}(p) \ . \end{aligned} \tag{29}$$

In Sutherland's form we have:

$$\begin{aligned} \frac{M^z}{L} &= \frac{1}{2}[1 - \int \mathrm{d}\nu \rho_{\rm h}(\nu) - 2\int \mathrm{d}u \rho_{\rm sp}(u)] \ ; \\ \frac{N}{L} &= 1 - \int \mathrm{d}\nu \rho_{\rm h}(\nu) \end{aligned} \tag{30}$$

with the same limits of integrations as in equations (25–28).

Apparently, the Bethe ansatz integral equations for densities of spin and charge rapidities do not depend on the coupling parameter A neither in Lai-Schlottmann's form, nor in Sutherland's form. This means that the distributions of charge and spin rapidities depend on the NNN interactions *indirectly* (*via* the limits of integrations which are determined from the equations for the dressed energies, see below).

3.2 Properties of the ground state

Let us treat the properties of the ground state of the STJC with the couplings. First consider Sutherland's representation. For $N = L$ the holons are absent, see equations (30). In this case of the half-filled band equations (27, 28) describe the Heisenberg spin $\frac{1}{2}$ chain, with nearest and NNN couplings (or two spin chains coupled with the zigzag-like interaction) [8]. The value of the chemical potential which corresponds to half-filling of the STJC with NNN interactions in the limit of zero magnetic field is equal to

$$\mu_{\rm s} = 2\ln 2 + 3A\zeta(3) \ . \tag{31}$$

The presence of the coupling A affects the value of the chemical potential, compared to its value for $A = 0$, $\mu = 2\ln 2$. Hence, if one fixes the chemical potential at its value for the STJC with the only nearest neighbor interactions and switches on a coupling between the NNN sites, the system will have less than one electron per site. This means that some number of holons will have negative energies (Dirac sea) – "spontaneous charge ordering". Then, according to equations (30), there is nonzero (spontaneous) magnetization in such a correlated electron system with NNN coupling. The limiting case of exactly one electron sitting at each site of two coupled correlated electron chains naturally coincides with the two-chain spin $\frac{1}{2}$ antiferromagnetic model, which has been studied previously [8].

The zero magnetic field behavior of the model with $N = L$ depends on the value of the coupling constant A (for simplicity we restrict ourselves to the case $A < 0$; $A > 0$ can be studied analogously). For $A > A_{\rm cr} = -4/\pi^2$

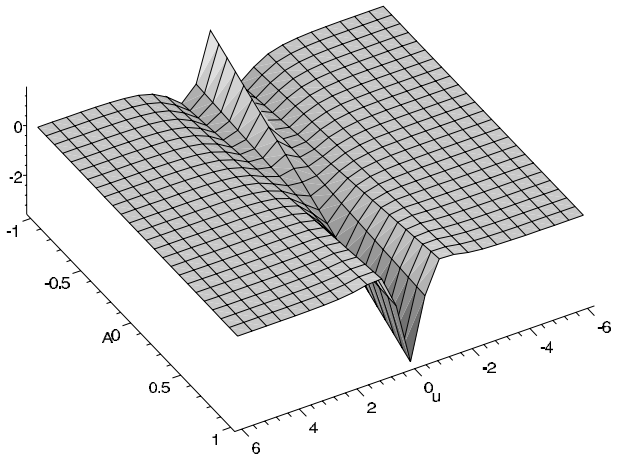

Fig. 1. The dressed energy of a spinon $\varepsilon_{\rm sp}$ for $H = 0$ as a function of the rapidity u and the coupling parameter A. Apparently at $A < A_{\rm cr}$ some dressed energies become positive for $H = 0$. One can see additional extrema for $A < A_1$. H plays the role of a chemical potential for spinons, changing the zero level.

($H = 0$) there is one Dirac sea for the low-lying excitations (spinons). The presence of the coupling between the spin chains (between the NNN in the representation of a single chain) renormalizes the dressed energies of spinons (hence, their Fermi velocities), but does not change the distribution of rapidities of spinons. However, if the absolute value of the coupling constant A is larger than the critical value, out of the mentioned domain, ($H = 0$) the eigenvalues of the energies of spinons are distributed in two Dirac seas symmetrically placed around 0. We can refer to this situation as for the onset of the band of "holes" of spinons in the vicinity of zero, see Figure 1. This means that there are four Fermi points (two for spinons and two for "holes") and two Fermi velocities. Hence the model reveals a phase transition (of second order) at that critical value of the coupling constant. Note that according to equations (30) the presence of "holes" in the distribution of the rapidities of spinons means *nonzero magnetization* of the model with NNN interactions even in the absence of an external magnetic field (spontaneous magnetization, *i.e.*, the system is in a *ferrimagnetic phase*). It also implies that if one decreased the magnetic field to negative values in this phase, a hysteresis could take place. Thus at zero magnetic field a first order phase transition takes place.

Let us consider the situation for our correlated electron model with one electron per site in an external magnetic field. The dressed energy of spinons (as well as the dispersion law for spinons, *i.e.*, the dependence of the dressed energy of spinon $\varepsilon_{\rm sp}$ on its quasimomentum) for the values of the coupling constant A larger in absolute value than some critical value $A_1 < 0$ displays one minimum (the spectrum is gapless and displays one maximum at $k = \pi/2$, k being the quasimomentum of spinon), see Figure 2a. However for $A < A_1$ there appear *two minima and one maximum* in the dressed energy (two maxima and one minimum at $k = \pi/2$, respectively, for the dispersion law of the spinon), see Figure 2c. Actually at $A = A_1$ the minimum of the dispersion law disappears and maxima merge into one at $k = \pi/2$, or, in other words, the maximum for $\varepsilon_{\rm sp}(u)$ disappears and two minima merge into one at $u = 0$, see Figure 2b. The zero slopes of the dispersion law correspond to the van Hove singularities of the bands of spinons (or their "holes" for the maximum of the dressed energy). For any value of the magnetic field for $A > A_1$ (Fig. 2a) and if the value of the magnetic field is less than some critical value $H_{\rm c}$, for $A_{\rm cr} < A < A_1$ (Fig. 2c) there is only one Dirac sea for spinons. On the other hand, for $H > H_{\rm c}$ and $A_{\rm cr} < A < A_1$ (Fig. 2c) there are *two Dirac seas for spinons*. Hence, there is a second order phase transition at $H = H_{\rm c}$, which pertains to the van Hove singularity of the empty band of "holes" of spinons (corresponding to the maximum in the dressed energy). This phase transition is between the *commensurate and incommensurate* magnetic phases, both with *gapless* spinon excitations, see Figure 3. At the value of the external magnetic field $H_{\rm s} = 4(1 + 8A)$ the system passes to the spin-saturation (ferromagnetic) phase at zero temperature. It is a second order phase transition between a phase with incommensurate correlations (due to the field induced magnetization) and the ferromagnetic phase. For $A > A_{\rm cr}$ the low-field phase is commensurate in the limit $H = 0$, for $A < A_{\rm cr}$ the phase is incommensurate even for $H = 0$. For $A < A_{\rm cr}$ (see Fig. 2e) the phase transition is between a ferrimagnetic (incommensurate) and a ferromagnetic phase, see Figure 3. In the ferromagnetic state spinon excitations are gapped. The point $A = A_1$ for $H = H_{\rm c}$ is a *tricritical* point (here $H_{\rm s} = H_{\rm c}$), see Figure 2d. At this point the singularities of the thermodynamic characteristics are proportional to $(H-H_{\rm c})^{1/4}$ (not to $(H-H_{\rm c})^{1/2}$ as for other points on the $H_{\rm c}$ line). It turns out that for $A < A_{\rm cr}$ there is no symmetry $H \to -H$, as it must be for a system with spontaneous magnetic ordering. Summarizing, there are three special lines in the ground state magnetic field behavior of the STJC with the NNN interactions (coupled two-chain STJC) with one electron per site: $H = H_{\rm s}$, which is the line of the second order phase transition to the ferromagnetic state, $H = 0$ for $A < A_{\rm cr}$, at which the first order phase transition takes place, and $H = H_{\rm c}$ for $A_{\rm cr} < A < A_1$, which is the line of the second order phase transition between commensurate and incommensurate magnetic states, see Figure 3.

The low temperature Sommerfeld theory manifests the usual linear T behavior of the specific heat (and finite zero-temperature magnetic susceptibility). At the lines of the phase transitions the van Hove singularities of empty bands of spinons produce $\sqrt{T}$ behavior. At the tricritical point the singularity is $T^{1/4}$.

3.3 Conformal behavior

Let us illustrate the above with the calculations of the asymptotics for the correlation functions of spinons. As a consequence of conformal invariance of (1+1)-dimensional quantum systems, the classification of universality classes is simple in terms of the central charge (conformal

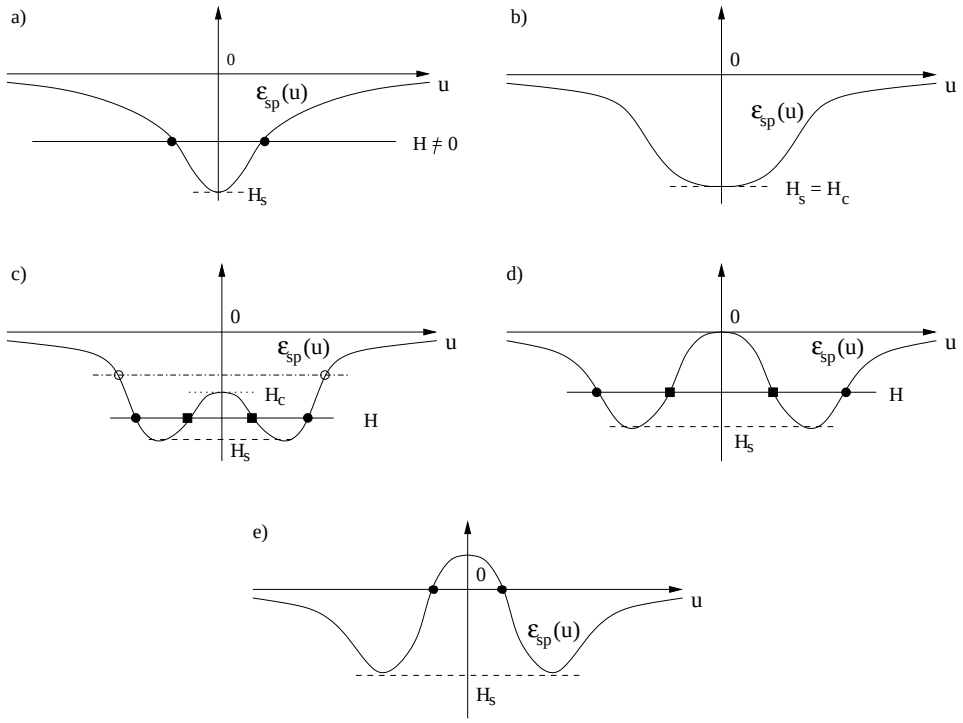

Fig. 2. The dressed energy of spinons $\varepsilon_{\rm sp}$ as a function of the rapidity u for different coupling parameters $A < 0$: $A_{\rm cr}, A_1 < A < 0$ (a), $A = A_1$ (b), $A_{\rm cr} < A < A_1$ (c), $A = A_{\rm cr}$ (d), $A < A_{\rm cr}$ (e). Depending on the value of the magnetic field H there is one Fermi sea in case (a), however up to two Fermi seas in case (c). The limiting case of a dressed energy function $\varepsilon_{\rm sp}(u)$ with just one minimum and that with two minima and one maximum is shown in (b). Here all derivatives of order 1 to 3 vanish at $u = 0$. In case (e) the coupling A is so strong that for all values of the field H (even $H = 0$) two Fermi seas are created. The limiting case to (c) is shown in (d). Filled circles and filled squares denote "old" and "new" Fermi points for $H \neq 0$. Notice that for the case (c) for $H < H_{\rm c}$ there are only two Fermi points (open circles).

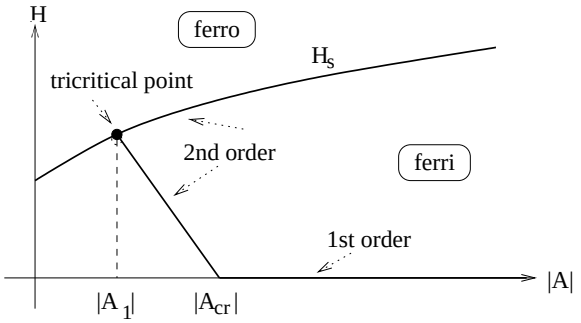

Fig. 3. Qualitative depiction of the ground state phase diagram in the magnetic field–coupling parameter plane $H - A$. The sections and limiting points $|A| < |A_1|$, $|A| = |A_1|$, $|A_1| < |A| < |A_{\rm cr}|$, $|A| = |A_{\rm cr}|$, $|A_{\rm cr}| < |A|$ correspond to cases (a–e) in Figure 2, respectively.

anomaly c) of the underlying Virasoro algebra [31]. The critical exponents in a conformally invariant theory are scaling dimensions of the operators within the quantum model. They can be calculated considering the finite-size (mesoscopic) corrections for energies and quasimomenta of the ground state and low-lying excited states. Conformal invariance formally requires all gapless excitations to have the same velocity (Lorentz invariance). The complete critical theory for systems with several gapless excitations with *different* Fermi velocities is usually given as a *semidirect product* of these independent Virasoro algebras [32]. Here we briefly sketch the procedure and write the results for the finite-size corrections to the energy, following the standard procedure [32]. One can see that for $A > A_1$ or for $A_{\rm cr} < A < A_1$, $H < H_{\rm c}$ the conformal limit of our STJC with the NNN interactions (coupled two-chain STJC) with one electron per site pertains to *one* level-1 Kac-Moody algebra (*one* Wess-Zumino-Novikov-Witten (WZNW) model of level 1 with the conformal anomaly $c = 1$). The finite-size correction to the energy is rather standard (*cf.* [32])

$$LE_{\rm fs} = -\frac{\pi}{6} v_{\rm F} + 2\pi v_{\rm F}(\Delta_{\rm l} + \Delta_{\rm r}) \ , \tag{32}$$

where $v_{\rm F}$ is the Fermi velocity of the spinon and the conformal dimensions Δ of primary operators are (note: the lower indices denote the conformal dimensions for right- and left-moving quasiparticles, at the right and left Fermi point $\pm B$, respectively):

$$2\Delta_{\rm l,r} = \left(\frac{\Delta M}{2z} \pm z\Delta D \right)^2 + 2n_{\rm l,r} \ , \tag{33}$$

where ΔM is an integer denoting the change of the number of particles induced by the primary operator, ΔD is an integer (half-integer) denoting the number of transfered particles from the right to the left Fermi point (backward scattering processes), $n_{\rm l,r}$ are the numbers of the particle-hole excitations of right- and left-movers. The values for the quantum numbers are restricted to $\Delta D = \Delta M/2$

(mod 1). The dressed charge $z = \xi(B)$ is the solution of the (standard) integral equation [32]

$$\xi(u) + \int_{(B)} \mathrm{d}v a_2(u-v)\xi(v) = 1 \ , \qquad (34)$$

taken at the limits of integration (these are the Fermi points, symmetric with respect to zero). In this phase there is only one region of integration over v. The dressed charge is a scalar. The behavior of our two-chain model in this phase in the conformal limit is rather standard [32]. The correlation functions decay asymptotically $\propto (x - \mathrm{i}v_{\mathrm{F}}t)^{-\Delta_{\mathrm{l}}}(x+\mathrm{i}v_{\mathrm{F}}t)^{-\Delta_{\mathrm{r}}}$. The choice of the appropriate quantum numbers of excitations ΔM, ΔD and $n_{\mathrm{l,r}}$ is determined for the leading asymptotics of correlators by taking the possible numbers with smallest exponents. Note that the dressed charge, and hence the exponents of the asymptotics do not depend on the coupling constant A in this phase.

On the other hand, for $A < A_{\mathrm{cr}}$ or for $A_{\mathrm{cr}} < A < A_1$, $H > H_{\mathrm{c}}$ the conformal limit of our STJC with the NNN interactions (coupled two-chain STJC) with one electron per site corresponds to the semidirect product of *two* level-1 Kac-Moody algebras, both with conformal anomalies $c = 1$, *i.e.*, to *two* WZNW models both of level 1. The Dirac seas (*i.e.*, the possible spinons with negative energies) are in the intervals $[-B^+, -B^-]$ and $[B^-, B^+]$ (minima in the distributions of rapidities). In fact the valley in the density distribution for "particles" and the maximum for "holes" are in one-to-one correspondence with the maxima and minimum of the dispersion law for spinons. The critical coupling constant A_{cr} or critical H_{c} in this language corresponds to the emergence of van Hove singularities of the band of "holes". The Fermi velocity of "particles" is $v_{\mathrm{F}}^{+} = (2\pi\rho_{\mathrm{sp}}(B^+))^{-1}\varepsilon'_{\mathrm{sp}}(u)|_{u=B^+}$, the Fermi velocity of "holes" is $v_{\mathrm{F}}^{-} = -(2\pi\rho_{\mathrm{sp}}(B^-))^{-1}\varepsilon'_{\mathrm{sp}}(u)|_{u=B^-}$. The finite-size corrections to the energy for this case are

$$LE_{\mathrm{fs}} = -\frac{\pi}{6}(v_{\mathrm{F}}^{+} + v_{\mathrm{F}}^{-}) + 2\pi\Big(v_{\mathrm{F}}^{+}(\Delta_{\mathrm{l}}^{+} + \Delta_{\mathrm{r}}^{+}) + v_{\mathrm{F}}^{-}(\Delta_{\mathrm{l}}^{-} + \Delta_{\mathrm{r}}^{-})\Big) \ , \qquad (35)$$

where the dispersion laws of "particles" and "holes" are linearized about the Fermi points for each Dirac sea. The conformal dimensions of primary operators are (upper indices denote Dirac seas; lower indices denote right and left Fermi points of each of these two Dirac seas:

$$2\Delta_{\mathrm{l,r}}^{\mp} = \left[\frac{(x_{-\pm}\Delta M^{+} - x_{+\pm}\Delta M^{-})}{2\det\hat{x}} \mp \frac{(z_{-\pm}\Delta D^{+} - z_{+\pm}\Delta D^{-})}{2\det\hat{z}}\right]^2 + 2n_{\mathrm{l,r}}^{\mp} \ , \qquad (36)$$

where the sign "minus" ("plus") between the terms in square brackets corresponds to right- (left-) movers. Here $\Delta M^{\pm}$ denotes the differences between the numbers of particles excited in the Dirac seas of spinons and "holes", labelled by upper indices. $\Delta D^{\pm}$ denote the numbers of backward scattering excitations, and $n_{\mathrm{l,r}}^{\pm}$ are the numbers of the particle-hole excitations for right- and left-movers of each Dirac sea (for spinons and "holes"). Note that $\Delta M^{\pm}$ and $\Delta D^{\pm}$ are *not independent*. Their values are restricted by the following connections: $\Delta M^{+} - \Delta M^{-} = \Delta M$, and $\Delta D^{+} - \Delta D^{-} = \Delta D$, where ΔM and ΔD determine in a standard way the changes of the total magnetization and the total momentum of the system, respectively, due to low-lying excitations. The same is true for the excitations which change the total magnetization of the system: there are only *two independent* of four such possible excitations. This is a direct consequence of the fact that *only one magnetic field* determines the filling of the Dirac seas for "particles" and "holes", or in other words, two Dirac seas for spinons.

The dressed charges $x_{ik}(B^k)$ and $z_{ik}(B^k)$ $(i,k = +,-)$ are matrices in this phase. Dressed charges measure the number of "bare" particles per dressed excitation. They can be expressed by using the solution of the integral equation [32,33]

$$f(u|B^{\pm}) + \left(\int_{-B^+}^{B^+} - \int_{-B^-}^{B^-}\right) a_2(u-v) f(v|B^{\pm}) = a_2(u - B^{\pm}) \ , \qquad (37)$$

with [32]

$$z_{ik}(B^k) = \delta_{i,k} + (-)^k \frac{1}{2}\Big(\int_{B^i}^{\infty} - \int_{-\infty}^{-B^i}\Big)\mathrm{d}v f(v|B^k)$$
$$x_{ik}(B^k) = \delta_{i,k} - (-)^k \int_{-B^i}^{B^i} \mathrm{d}v f(v|B^k) \ . \qquad (38)$$

Note that the dressed charges depend indirectly on the values of the intra-chain (or the next to the nearest neighbour) coupling constant A, and H only *via* the limits of integrations. In the first order approximation one can write the solutions as $x_{ik}(B^k) \approx \delta_{i,k} - (-)^k \int_{-B^i}^{B^i} \mathrm{d}v a_2(v - B^k) + \ldots$ and $z_{ik}(B^k) \approx \delta_{i,k} + (-)^k(1/2)(\int_{B^i}^{\infty} - \int_{-\infty}^{-B^i})\mathrm{d}v a_2(u - B^k) + \ldots$ The slopes of the dressed energies of "particles" and "holes" at Fermi points of the Dirac seas (Fermi velocities) differ in general from each other. Therefore we have a semidirect product of two algebras. Hence, in those regions of A and H the dressed charges are 2×2 matrices. It means that the conformal limit of our STJC with the NNN couplings (two coupled STJCs) with one electron per site corresponds to *one* or *two* WZNW theories depending on the values of the intra-chain coupling and magnetic field. At the critical line A_{cr} (or at H_{c}) the Dirac sea of "holes" disappears as well as the components of the dressed charge matrix $\hat{x}$ (with the square root singularities of the critical exponents for correlation functions). Note that the dressed charge z becomes $z = (2x)^{-1}$ at the phase transition line H_{c}. This corresponds to the disappearance of one of the WZNW models. Unfortunately it is impossible to obtain an analytic solution to equations (37) in a closed form for the finite NNN (intra-chain) coupling A. Naturally, in the limiting case of purely nearest neighbour couplings (single STJC),

$A = 0$, the solutions of equations (34, 37, 38) coincide with the well-known ones [32]. The correlation functions of the STJC with NNN couplings (two coupled STJCs) with one electron per site decay algebraically in those phases $\propto (x - \mathrm{i}v_{\mathrm{F}}^{+}t)^{-\Delta_{\mathrm{l}}^{+}}(x - \mathrm{i}v_{\mathrm{F}}^{-}t)^{-\Delta_{\mathrm{l}}^{-}}(x + \mathrm{i}v_{\mathrm{F}}^{+}t)^{-\Delta_{\mathrm{r}}^{+}}(x + \mathrm{i}v_{\mathrm{F}}^{-}t)^{-\Delta_{\mathrm{r}}^{-}}$ with the minimal exponents of possible quantum numbers of excitations $\Delta M^{\pm}$, $\Delta D^{\pm}$ and $n_{\mathrm{l,r}}^{\pm}$. We point out once again that the same magnetic field plays the role of an effective "chemical potential" for the Dirac seas of "particles" and "holes", or spinons of both Dirac seas. Hence this choice of "minimal quantum numbers" is constrained. This drastically differs from the standard case of two level-1 WZNW models for charge and spin excitations in the STJC out of half-filling, because there the Dirac seas for spinons and charge excitations are governed by different Lagrange multipliers, namely, the magnetic field and chemical potential, respectively. The above conformal analysis is valid for the system out of phase transition lines (*i.e.*, $H \neq 0, H_{\mathrm{c}}, H_{\mathrm{s}}$ for the above mentioned regions of A). At the lines of phase transitions one of the Fermi velocities becomes zero and the appropriate finite size corrections to the ground state energy, which we discussed, disappear.

3.4 Charge sector

Now we consider the case of only charge excitations are present in the ground state. This situation can be studied more naturally in the Lai-Schlottmann's formulation. Suppose the band of spin excitations (unbound electrons) is empty. This can be achieved by the appropriate choice of H. Hence all the electrons are bound into spin-singlet pairs. We know that the creation of holes in the Dirac sea for pairs starts from zero values of rapidities [24]. That is why we can study the behavior of the dressed energies of the holes of pairs instead of the behavior of the pairs themselves with the replacement $\varepsilon_{\mathrm{c}}(\lambda) \to -\tilde{\varepsilon}_{\mathrm{c}}(\lambda)$ in equations (26). In this form the equation for the dressed energies of holes of pairs equations (26) coincides (up to the renormalization of the driving term) with the equation for the dressed energies of spinons equations (28). The changes in the driving term are as follows – one has to replace H by 2μ (pairs) and $a_1(u)$ by $a_2(\lambda)$. Hence, up to these (quantitative) changes the analysis of the behavior of the charge excitations in this situation is similar to the previous one for the spin sector. This implies that for $A > A_1^{\mathrm{c}}$ ($A_1^{\mathrm{c}} < 0$) there is one Dirac sea for the holes of singlet pairs. This phase corresponds in the conformal limit to one level-1 WZNW theory. Here the dressed charge for pairs is scalar. The exponents for the asymptotics of correlation functions and the dressed charge do not depend on the coupling constant in this phase. For $A_1^{\mathrm{c}} > A > A_{\mathrm{cr}}^{\mathrm{c}} = -3\zeta(3)/2\ln 2$ our system can be either in the commensurate gapless phase or in the incommensurate gapless phase, depending on the band filling. This means that at some filling there is a phase transition between the commensurate and incommensurate phases (in other words, if one applies an external voltage, there is a critical value of that voltage, which corresponds to a second order phase transition). Finally, for $A < A_{\mathrm{cr}}^{\mathrm{c}}$ the system is in a charged incommensurate phase. Here, the soft modes of the elementary charge excitation take values incommensurate with the Brillouin zone boundary. As a consequence, two-point correlations show oscillations that are not commensurate with the underlying lattice. We can consider an external applied voltage as the Lagrange multiplier, controlling the filling of the Dirac sea(s) for pairs. At the external voltage $\mu = 0$ (zero chemical potential) we expect spontaneous appearance of holes of pairs (*i.e.*, "spontaneous charge ordering") for $A < A_{\mathrm{cr}}^{\mathrm{c}}$ with a hysteresis phenomenon (for negative values of μ). Thus this phase transition is of first order. There are two Dirac seas for the charge excitations (holes of pairs) in this phase, and the conformal limit corresponds to the semidirect product of two level-1 WZNW theories for charge excitations. At the critical value of coupling constant $A_{\mathrm{cr}}^{\mathrm{c}}$ the second Dirac sea is closed with the van Hove singularities (second order phase transition). The same happens for $A_{\mathrm{cr}}^{\mathrm{c}} < A < A_1^{\mathrm{c}}$ at the critical value of an external voltage. Note that the above mentioned choice of H excluding spin carrying excitations of unbound electrons corresponds to negative values of the magnetic field for $A < A_{\mathrm{cr}}$. This means that for the zero field case the state of the STJC with the NNN interactions (coupled two-chain model) has nonzero magnetization (hence the zero temperature susceptibility diverges). For the nonzero values of the applied voltage the analysis of the behavior of the low-lying charge excitations is similar to the previous one for the spin excitations in the nonzero external magnetic field. At the critical value of the applied voltage $\mu_{\mathrm{s}} = 2\ln 2 + 3A\zeta(3)$ the second order phase transition takes place. It corresponds to the van Hove singularity of the filled band of pairs (*i.e.*, the system has one electron per site). It turns out that the above conformal analysis also corresponds to the case of the magnetic field and applied potential differing from their critical values. At the lines of phase transitions the appropriate Fermi velocities become zero and the appropriate finite size corrections disappear. Again, the low temperature Sommerfeld expansion manifests the usual linear T behavior of the specific heat (and finite zero-temperature charge susceptibility). At the lines of the phase transition the van Hove singularity of the empty bands produces $\sqrt{T}$ behavior. At the tricritical point we expect $T^{1/4}$ behavior as before.

3.5 Mixed spin and charge sectors

Now let us concentrate on the situation with Dirac seas for spin and charge excitations present in the ground state. The situation is very complicated. In principle one has to solve two coupled integral equations. There the integrations are over intervals with negative dressed energies of low-lying spin and charge excitations. There are, generally speaking, two (one for each kind of excitations), three or four Dirac seas, depending on the values of the coupling constant, external magnetic field and applied potential. Those situations pertain in the conformal limit to the semidirect products of two, three and four Gaussians (level-1 Kac-Moody algebras with central charges $c = 1$).

Note that only two kinds of all those low-lying conformal excitations are independent, because the filling of all Dirac seas for charges is governed by only one external voltage, and the filling of the Dirac seas for spin excitations is governed by one external magnetic field. At zero values of the external magnetic field and/or applied voltage the system can undergo first order phase transitions. The changes of the values of the external magnetic field or applied voltage drive the system to the ferromagnetic phase or to a phase with exactly one electron per site with a second order phase transitions at the critical lines. It is, unfortunately, impossible to write down the analytic expressions for the critical values $H_{\rm s}$ and $\mu_{\rm s}$. However, for some important limiting cases we can obtain analytic answers in closed form. For example, for the situation of the magnetic field value being much smaller than the bandwidth for holons (charge excitations), which is natural from the viewpoint of the realization in experiment, we can calculate the value of $\mu_{\rm s}$

$$\mu_{\rm s} = 2\ln 2 + 3A\zeta(3) + \frac{H^2}{4\pi^2} + \dots \tag{39}$$

So, the critical value of the chemical potential depends in general on the value of the external magnetic field. We can also calculate the value of the transition field $H_{\rm s}$ to the ferromagnetic phase. It corresponds to the energy of spinon at maxima and zero population of the spinon's Dirac sea. Equations (28) yield

$$H_{\rm s} \approx X + 2B(\mu + 2X)\ , \tag{40}$$

where ($B \ll 1$) is the Fermi point for spinons

$$B^2 = 2\frac{\mu - 2\ln 2 - 3A\zeta(3)}{3\zeta(3) + 45A\zeta(5)}\ , \tag{41}$$

and $X = 4(1 + 8A)$. These equations mean that the critical value of the magnetic field depends on the external voltage, or on the band filling. For zero values of the external magnetic field and/or external applied voltage we expect the spontaneous filling of the band for the holes of charge excitations and/or spontaneous magnetization for large enough values of the coupling constant (at least for $A < A_{\rm cr}$).

4 Summarizing remarks

In this paper we have proposed the exactly solvable model of the supersymmetric $t-J$ model with nearest and next nearest neighbor couplings, or in other words, two $t-J$ chains coupled by zigzag-like interactions and also by four-site couplings. This study was motivated for instance by recent experiments on vanadates and low dimensional magnetic Jahn-Teller compounds. In dependence on the coupling constant we have obtained a rich variety of ground state phases of the system. The next-nearest neighbor coupling (intrachain coupling in the two-chain language) of correlated electrons causes the onset of additional Dirac seas for the charge and/or spin excitations. These Dirac seas change the low-energy conformal behavior of the model which strongly depends on the values of the additional coupling, external magnetic field and applied voltage. We identify the new phases which appear due to the next-nearest neighbor interactions (intra-chain coupling for two chains) as incommensurate phases, because in those phases the soft modes of charge and/or spin excitations take values incommensurate with the Brillouin zone boundary. Hence there are incommensurate oscillations of two-point functions of spin and/or charge operators. The ground state (quantum) phase transitions between the commensurate and incommensurate phases are of second order. Moreover, the coupling between the correlated electron chains in the incommensurate phases causes the onsets of the spontaneous magnetization and/or spontaneous filling of the Dirac sea for charge excitations ("charge ordering"). The appearance of these spontaneous values implies the first order phase transitions with hysteresis phenomena. To the best of our knowledge, the presented model is the first integrable model which exhibits a variety of commensurate and incommensurate charge and magnetic phases together with onsets of spontaneous "charge ordering" and magnetization.

The effects of the coupling between the correlated electron chains, which we have studied in this paper, are of the same nature as the incommensurate phases, which exist in a chain of noninteracting electrons (fermions with spin) with nearest and next-nearest neighbour hoppings. The main difference lies in the nature of the Dirac seas: In the noninteracting case the Dirac seas appear for the spin-up and -down *noninteracting electrons*, while in our case of the correlated electron model Dirac seas pertain to spin and charge *low-lying excitations*. However the nature of phase transitions between commensurate and incommensurate phases are similar – van Hove singularities of additional Dirac seas. We also point out that strong correlations between electrons in the $t-J$ model with next-nearest neighbor couplings cause the emergences of simultaneous "charge ordering" and spontaneous magnetization, which are naturally absent in the system of noninteracting electrons with hopping between nearest and next-nearest neighboring sites of the lattice.

A.A.Z. acknowledges financial support from *Deutsche Forschungsgemeinschaft* at the earlier stages of the work and thanks G.I. Japaridze for helpful discussions. A.K. acknowledges financial support by *Deutsche Forschungsgemeinschaft* under grant No. Kl 645/3-3.

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